

Finite groups in which SS-permutability is a transitive relation*

Xiaoyu Chen, Wenbin Guo

Department of Mathematics, University of Science and Technology of China,
Hefei 230026, P. R. China

E-mail: jelly@mail.ustc.edu.cn, wbguo@ustc.edu.cn

Abstract

A subgroup H of a finite group G is said to be SS-permutable in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K . A finite group G is called an SST-group if SS-permutability is a transitive relation. The structure of SST-groups is investigated in this paper.

1 Introduction

Throughout this paper, all groups considered are finite. For a group G , let $\pi(G)$ denote the set of prime divisors of $|G|$, and $|G|_p$ denote the order of a Sylow p -subgroup of G .

A subgroup H of a group G is said to be *permutable* (resp. *S-permutable*) in G if H permutes with all the subgroups (resp. Sylow subgroups) of G . A group G is called a *T-group* (resp. *PT-group*, *PST-group*) if normality (resp. permutability, S-permutability) is a transitive relation, that is, if H and K are subgroups of G such that H is normal (resp. permutable, S-permutable) in K and K is normal (resp. permutable, S-permutable) in G , then H is normal (resp. permutable, S-permutable) in G .

By Kegel's result [15], a group G is a PST-group if and only if every subnormal subgroup of G is S-permutable in G . Agrawal [1] showed that a group G is a solvable PST-group if and only if the nilpotent residual L of G is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms. In particular, a solvable PST-group is supersolvable. The structure of PST-groups has been investigated by many authors, see for example [2–6, 8].

*Research is supported by a NNSF grant of China (grant #11071229) and Research Fund for the Doctoral Program of Higher Education of China (Grant 20113402110036).

Keywords: SS-permutability, S-semipermutability, PST-groups, BT-groups, SST-groups.
Mathematics Subject Classification (2000): 20D10, 20D20, 20D35.

A subgroup H of a group G is said to be *semipermutable* (resp. *S-semipermutable*) [10] in G if H permutes with every subgroup (resp. Sylow subgroup) X of G such that $(|H|, |X|) = 1$. A group G is called a *BT-group* (resp. *SBT-group*) [21] if semipermutability (resp. S-semipermutability) is a transitive relation. The following result was established by Wang et al. in [21].

Theorem 1.1 [21, Theorem 3.1]. Let G be a group. Then the following statements are equivalent:

- (1) G is a solvable BT-group.
- (2) G is a solvable SBT-group.
- (3) Every subgroup of G is semipermutable in G .
- (4) Every subgroup of G is S-semipermutable in G .
- (5) Every subgroup of G of prime power order is semipermutable in G .
- (6) Every subgroup of G of prime power order is S-semipermutable in G .
- (7) G is a solvable PST-group and if p and q are distinct primes not dividing the order of L , where L is the nilpotent residual of G , then $[G_p, G_q] = 1$ for every $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$.

Recall that a subgroup H of a group G is said to be *SS-permutable* or *SS-quasinormal* in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K . In this case, K is called an SS-permutable supplement of H in G . This embedding property was introduced by Li et al. in [16], and they called these subgroups SS-quasinormal subgroups. In this paper, we use the term SS-permutable subgroups for such subgroups. It is easy to see that every SS-permutable subgroup of a group G is S-semipermutable in G (see below Lemma 2.1(4)). However, the converse does not hold in general as the following example shows.

Example 1.2. Let $G = \langle x, y, z, w \mid x^3 = y^3 = z^2 = w^2 = 1, [x, y] = [z, w] = 1, x^z = x^{-1}, y^w = y^{-1}, x^w = x^{-1}, y^z = xy \rangle$ and $H = \langle y, w \rangle$. Obviously, H is S-semipermutable in G . Now assume that H is SS-permutable in G with an SS-permutable supplement K . Let K_2 be a Sylow 2-subgroup of K . Then by definition, H permutes with K_2 . It is clear that $\langle y \rangle \trianglelefteq H \trianglelefteq HK_2$, and so $HK_2 \leq N_G(\langle y \rangle)$. This implies that $z \in N_G(\langle y \rangle)$, which is contrary to our assumption. Therefore, H is not SS-permutable in G .

We say that a subgroup H of a group G is *NSS-permutable* in G if H has a normal supplement K in G such that H permutes with every Sylow subgroup of K . In this case, K is called an NSS-permutable supplement of H in G . Note that every NSS-permutable subgroup of a group G is SS-permutable in G , but the next example illustrates that the converse is not true.

Example 1.3. Let $G = A_5$ and $H = A_4$, where A_5 and A_4 denote the alternating group of

degree 5 and 4, respectively. Clearly $K \in Syl_5(G)$ is an SS-permutable supplement of H in G . This shows that H is SS-permutable in G . If H is NSS-permutable in G , then G is the only NSS-permutable supplement of H in G . This induces that H is S-permutable in G , and so H is subnormal in G by [15, Theorem 1], which is impossible. Hence H is not NSS-permutable in G .

We say that a group G is an *SST-group* (resp. *NSST-group*) if SS-permutability (resp. NSS-permutability) is a transitive relation.

Our purpose in this paper is to establish several characterizations of SST-groups and NSST-groups. We arrive at the following results.

Note that a group is said to be an *SC-group* if all its chief factors are simple. Clearly, the class of all supersolvable groups coincides with the class of all solvable SC-groups.

Theorem A. Let G be an SST-group (resp. NSST-group). Then G is an SC-group.

Theorem B. Let G be a group. Then the following statements are equivalent:

- (1) G is solvable and every subnormal subgroup of G is SS-permutable in G .
- (2) G is solvable and every subnormal subgroup of G is NSS-permutable in G .
- (3) Every subgroup of $F^*(G)$ is SS-permutable in G .
- (4) Every subgroup of $F^*(G)$ is NSS-permutable in G .
- (5) G is a solvable PST-group.

In Theorem B above, $F^*(G)$ denotes the generalized Fitting subgroup of G , that is, the product of all normal quasinilpotent subgroups of G . The readers may refer to [14, Chapter X] for details.

Theorem C. Let G be a group. Then the following statements are equivalent:

- (1) Whenever $H \leq K$ are two p -subgroups of G with $p \in \pi(G)$, H is SS-permutable in $N_G(K)$.
- (2) Whenever $H \leq K$ are two p -subgroups of G with $p \in \pi(G)$, H is NSS-permutable in $N_G(K)$.
- (3) G is a solvable PST-group.

Theorem D. Let G be a solvable group. Then the following statements are equivalent:

- (1) G is an SST-group.
- (2) G is an NSST-group.
- (3) Every subgroup of G is SS-permutable in G .
- (4) Every subgroup of G is NSS-permutable in G .
- (5) Every subgroup of G of prime power order is SS-permutable in G .
- (6) Every subgroup of G of prime power order is NSS-permutable in G .

- (7) Every cyclic subgroup of G of prime power order is SS-permutable in G .
- (8) Every cyclic subgroup of G of prime power order is NSS-permutable in G .

Corollary 1.4. A solvable SST-group G is a BT-group.

The following example illustrates that a solvable BT-group is not necessarily a solvable SST-group.

Example 1.5. Let $G = \langle x, y \mid x^5 = y^4 = 1, x^y = x^2 \rangle$. Note that the nilpotent residual of G is $\langle x \rangle \in Syl_5(G)$. By Theorem 1.1, G is a solvable BT-group. Put $H = \langle y \rangle$ and $L = \langle y^2 \rangle$. Suppose that L is SS-permutable in G . Then G is the unique supplement of L in G . It follows that L is S-permutable in G , and thus $L \leq O_2(G)$ by [15, Theorem 1]. This implies that either $O_2(G) = H$ or $O_2(G) = L$. It is easy to verify that $y^x = yx^{-1}$ and $(y^2)^x = y^2x^2$, and so $H \not\leq G$ and $L \not\leq G$, a contradiction. Hence L is not SS-permutable in G . By Theorem D, G is not a solvable SST-group.

Corollary 1.6. The class of all solvable SST-groups is closed under taking subgroups and epimorphic images.

Corollary 1.7. Let G be a solvable group. Then the following statements are equivalent:

- (1) G is an SST-group.
- (2) Every subgroup of G is either SS-permutable or abnormal in G .
- (3) Every subgroup of G is either NSS-permutable or abnormal in G .

Let G be a solvable PST-group. Let L be the nilpotent residual of G and D be a system normalizer of G . Then L is a normal abelian Hall subgroup of G upon which D acts by conjugation as power automorphisms. Hence by [19, (9.2.7)], D is a Hall subgroup of G and $G = L \rtimes D$. Note that all the system normalizers of G are conjugate in G .

Recall that a group G is said to be complemented if every subgroup of G has a complement in G . In [13], Hall proved that the class of all complemented groups is exactly the class of all supersolvable groups with elementary abelian Sylow subgroups.

Theorem E. Let G be a solvable SST-group with the nilpotent residual L and a system normalizer D . Then the Frattini group $\Phi(G) = \Phi(L) \times \Phi(D)$. Furthermore, $G/\Phi(G)$ is a complemented group.

Theorem F. Let G be a solvable BT-group with the nilpotent residual L . Then the following statements are equivalent:

- (1) G is an SST-group.
- (2) For every p -subgroup P of G with $p \in \pi(G) \setminus \pi(L)$, G has a p -subgroup K_p such that $PK_p \in Syl_p(G)$ and $[P, \langle K_p^L \rangle] \leq O_p(G)$.

(3) For every p -element x of G with $p \in \pi(G) \setminus \pi(L)$, G has a p -subgroup K_p such that $\langle x \rangle K_p \in \text{Syl}_p(G)$ and $[x, \langle K_p^L \rangle] \leq O_p(G)$.

It was established in [11] that $G_1 \times G_2 \times \cdots \times G_n$ ($n \geq 2$) is a solvable PST-group if and only if G_i is a solvable PST-group with the nilpotent residual L_i for $1 \leq i \leq n$ and $(|L_i|, |G_j|) = 1$ for $1 \leq i, j \leq n$ and $i \neq j$. Now we continue the study for solvable BT-groups and solvable SST-groups.

Theorem G. $G_1 \times G_2 \times \cdots \times G_n$ ($n \geq 2$) is a solvable BT-group if and only if G_i is a solvable BT-group with the nilpotent residual L_i for $1 \leq i \leq n$ and $(|L_i|, |G_j|) = 1$ for $1 \leq i, j \leq n$ and $i \neq j$.

Theorem H. $G_1 \times G_2 \times \cdots \times G_n$ ($n \geq 2$) is a solvable SST-group if G_i is a solvable SST-group and $(|G_i|, |G_j|) = 1$ for $1 \leq i, j \leq n$ and $i \neq j$.

The next example shows that Theorem G does not hold for solvable SST-groups.

Example 1.8. Let $G = \langle x, y, z, w \mid x^3 = y^5 = z^2 = w^2 = 1, [x, y] = [x, w] = [z, y] = [z, w] = 1, x^z = x^{-1}, y^w = y^{-1} \rangle$, $G_1 = \langle x, z \rangle$ and $G_2 = \langle y, w \rangle$. It is clear that G_1 and G_2 are solvable SST-groups and $L_1 = \langle x \rangle$ and $L_2 = \langle y \rangle$ are the nilpotent residuals of G_1 and G_2 , respectively. Put $H = \langle zw \rangle$. It is easy to verify that G , $\langle x, y, z \rangle$ and $\langle x, y, w \rangle$ are not SS-permutable supplements of H in G . Hence H is not SS-permutable in G . Then by Theorem D, G is not a solvable SST-group.

In [9], J. C. Beidleman et al. proved that a group G is a solvable PST-group if and only if G has a normal subgroup N such that N and G/N'' are solvable PST-groups. Analogously, we have the following result.

Theorem I. A group G is a solvable SST-group (resp. BT-group) if and only if G has a normal subgroup N such that N is a solvable PST-group and G/N'' is a solvable SST-group (resp. BT-group).

2 Preliminaries

Lemma 2.1. Suppose that a subgroup H of a group G is SS-permutable (resp. NSS-permutable) in G with an SS-permutable supplement (resp. NSS-permutable supplement) K , $L \leq G$ and $N \trianglelefteq G$. Then:

- (1) If $H \leq L$, then H is SS-permutable (resp. NSS-permutable) in L .
- (2) HN/N is SS-permutable (resp. NSS-permutable) in G/N .
- (3) If $N \leq L$ and L/N is SS-permutable (resp. NSS-permutable) in G/N , then L is SS-permutable (resp. NSS-permutable) in G .

(4) H is S-semipermutable in G .

(5) If $H \leq F(G)$, then H is S-permutable in G .

(6) Every conjugate of K in G is an SS-permutable supplement (resp. NSS-permutable supplement) of H in G .

(7) If N is nilpotent, then NK is an SS-permutable supplement (resp. NSS-permutable supplement) of H in G .

(8) If H is a p -subgroup of G with $p \in \pi(G)$, then $HK_p \in Syl_p(G)$ for every $K_p \in Syl_p(K)$.

Proof. We only prove the statements for SS-permutable subgroups. For NSS-permutable subgroups, the statements can be handled similarly.

(1)-(3) See [16, Lemma 2.1].

(4) By definition, H permutes with every Sylow subgroup of K . Let X be a Sylow subgroup of G such that $(|H|, |X|) = 1$. Then there exists an element $h \in H$ such that $X^h \leq K$. It follows that $HX^h = X^hH$, and so $HX = XH$. Hence H is S-semipermutable in G .

(5) See [16, Lemma 2.2].

(6) The statement is obvious.

(7) For every prime $p \in \pi(K)$, there exist $N_p \in Syl_p(N)$ and $K_p \in Syl_p(K)$ such that $N_pK_p \in Syl_p(NK)$. Note that $N_p \trianglelefteq G$. Then for any Sylow p -subgroup P of NK , NK has an element $g = kh$ for some $k \in K$ and $h \in H$ such that $P = (N_pK_p)^g = (N_pK_p^k)^h$. Since K is an SS-permutable supplement of H in G , we have that $H(N_pK_p^k) = (N_pK_p^k)H$, and so $HP = H(N_pK_p^k)^h = (N_pK_p^k)^hH = PH$. Therefore, NK is an SS-permutable supplement of H in G .

(8) Evidently, K has an element k such that $HK_p^k \in Syl_p(G)$. Note that $H \cap K$ is S-permutable in K , and so it is subnormal in K by [15, Theorem 1]. Hence $H \cap K \leq O_p(K)$. This implies that $H \cap K = H \cap K_p = H \cap K_p^k$. It follows that $|G|_p = |HK_p^k| = |H||K_p^k|/|H \cap K_p^k| = |H||K_p|/|H \cap K_p| = |HK_p|$. Thus $HK_p \in Syl_p(G)$.

Lemma 2.2. [7, Lemma 2.1.3] Let p be a prime and N a normal p -subgroup of a group G . Then all subgroups of N are S-permutable in G if and only if all chief factors of G below N are cyclic and G -isomorphic when regarded as modules over G .

Recall that a subgroup H of a group G is said to be τ -quasinormal [17] in G if $HG_p = G_pH$ for every $G_p \in Syl_p(G)$ such that $(|H|, p) = 1$ and $(|H|, |G_p^G|) \neq 1$. It is easy to see that every SS-permutable subgroup of G is τ -quasinormal in G by Lemma 2.1(4).

Lemma 2.3. [18, Theorem 1.2] Let G be a group. Then every subgroup of $F^*(G)$ is τ -quasinormal in G if and only if G is a solvable PST-group.

Lemma 2.4. A solvable group G is a BT-group if and only if every cyclic subgroup of G of prime power order is S-semipermutable in G .

Proof. By Theorem 1.1, we only need to prove the sufficiency. Assume that every cyclic subgroup of G of prime power order is S-semipermutable in G . Let H be a p -subgroup of G with $p \in \pi(G)$. Then for every element $h \in H$ and every Sylow q -subgroup Q of G , we have that $\langle h \rangle Q = Q \langle h \rangle$. Consequently, we have that $HQ = QH$, and so H is S-semipermutable in G . This shows that every subgroup of G of prime power order is S-semipermutable in G . Hence by Theorem 1.1, G is a BT-group.

Lemma 2.5. Let G be a nilpotent-by-abelian group and $H \leq G$. Then H is SS-permutable in G if and only if H is NSS-permutable in G .

Proof. The sufficiency is clear. We only need to prove the necessity. Since G is nilpotent-by-abelian, $G/F(G)$ is abelian. Suppose that H is SS-permutable in G with an SS-permutable supplement K . Then by Lemma 2.1(7), $F(G)K$ is also an SS-permutable supplement of H in G . As $F(G)K \trianglelefteq G$, we have that $F(G)K$ is also an NSS-permutable supplement of H in G . Hence H is NSS-permutable in G .

Lemma 2.6. Let T and S are SS-permutable (resp. NSS-permutable) in a solvable group G such that $(|T|, |S|) = 1$. Then $\langle T, S \rangle$ is SS-permutable (resp. NSS-permutable) in G . The solvability assumption is not necessary if T and S are subgroups of prime power order.

Proof. We only prove the lemma for SS-permutable subgroups. Let K_1 and K_2 be SS-permutable supplements of T and S in G , respectively. Note that G is solvable. By Lemma 2.1(6), without loss of generality, we may assume that $S \leq K_1$ and $T \leq K_2$. Then $TS(K_1 \cap K_2) = TK_1 = G$. This means that $K_1 \cap K_2$ is a supplement of $\langle T, S \rangle$ in G . For any Sylow p -subgroup P of $K_1 \cap K_2$ with $p \in \pi(K_1 \cap K_2)$, there exist a Sylow p -subgroup K_{1p} of K_1 and a Sylow p -subgroup K_{2p} of K_2 such that $P = K_{1p} \cap K_{2p}$. Since $T(K_{1p} \cap K_2) = (K_{1p} \cap K_2)T$ and $S(K_1 \cap K_{2p}) = (K_1 \cap K_{2p})S$, we have that $\langle T, S \rangle P = P \langle T, S \rangle$. This shows that $K_1 \cap K_2$ is an SS-permutable supplement of $\langle T, S \rangle$ in G . Thus $\langle T, S \rangle$ is SS-permutable in G .

Lemma 2.7. Let G be a solvable PST-group with the nilpotent residual L and a system normalizer D . Then:

(1) $D' \trianglelefteq G$.

(2) If H is an SS-permutable p -subgroup of G with an SS-permutable supplement K , where $p \in \pi(G)$, then $[H, K_p] \leq O_p(G)$ for every $K_p \in Syl_p(K)$.

Proof. (1) Note that for every element $l \in L$, $\langle l \rangle$ is normal in G . Then $G/C_G(l)$ is abelian, and so $D' \leq G' \leq C_G(l)$. It follows that $D' \leq C_G(L)$. Hence $D' \trianglelefteq G$ for $G = L \rtimes D$.

(2) If $p \in \pi(L)$, then it is evident. Now assume that $p \in \pi(G) \setminus \pi(L)$. For every $K_p \in Syl_p(K)$, it follows from Lemma 2.1(8) that G has a Sylow p -subgroup G_p such that $G_p = HK_p$. By (1), $G_p' \trianglelefteq G$. Hence $[H, K_p] \leq G_p' \leq O_p(G)$.

Lemma 2.8. Let G be a solvable BT-group with the nilpotent residual L . Suppose that H is a p -subgroup of G with $p \in \pi(G) \setminus \pi(L)$. Then H is SS-permutable in G if G has a p -subgroup K_p such that $HK_p \in \text{Syl}_p(G)$ and $[H, \langle K_p^L \rangle] \leq O_p(G)$.

Proof. Let D be a system normalizer of G containing HK_p and $D_{p'}$ be the Hall p' -subgroup of D . Then we claim that $K = O_p(G)K_pD_{p'}L$ is an SS-permutable supplement of H in G . In fact, it is clear that $G = HK$. By Theorem 1.1, if $q \in \pi(K)$ and $q \neq p$, then H permutes with every Sylow q -subgroup of K . Note that $D_{p'} \leq C_G(O_p(G)K_p)$. Then for every element $g \in K$, we have that $[H, O_p(G)K_p^g] \leq O_p(G)$. This deduces that $HO_p(G)K_p^g = O_p(G)K_p^gH$. Hence H permutes with every Sylow p -subgroup of K . Thus the claim holds, and consequently, H is SS-permutable in G .

3 Proof of the Theorems

Proof of Theorem A. We only prove the theorem for SST-groups. For NSST-groups, the theorem can be proved similarly. Let N be a minimal normal subgroup of G . By Lemma 2.1(2) and (3), G/N is also an SST-group. By induction, G/N is an SC-group. Now we only need to prove that N is simple. Since G is an SST-group, all subnormal subgroups of G are SS-permutable in G . It follows that all subnormal subgroups of N are SS-permutable in G . If N is abelian, then N is a normal p -subgroup of G for some $p \in \pi(G)$. By Lemma 2.1(5), all subgroups of N are S-permutable in G . By Lemma 2.2, N is cyclic, and so N is simple. Now assume that N is non-abelian. Then $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is isomorphic to a non-abelian simple group S for $1 \leq i \leq n$. Note that N_1 is SS-permutable in G . Let K be an SS-permutable supplement of N_1 in G . Then $N_1K_p = K_pN_1$ for every $K_p \in \text{Syl}_p(K)$. Since $N_1^g \trianglelefteq N$ for every element $g \in G$, we have that $\langle N_1^{K_p} \rangle \trianglelefteq N$. By [12, A, Proposition 4.13(b)], $\langle N_1^{K_p} \rangle$ is a direct product of a subset of N_i ($1 \leq i \leq n$). Obviously, $\langle N_1^{K_p} \rangle / N_1 \leq N_1K_p / N_1$ is a p -group. This deduces that $\langle N_1^{K_p} \rangle = N_1$, and thereby $K_p \leq N_G(N_1)$. It follows that $K \leq N_G(N_1)$, and thus $N_1 \trianglelefteq G$. Hence $N = N_1$ is simple. This completes the proof of Theorem A.

Proof of Theorem B. Note that every NSS-permutable subgroup of a group G is SS-permutable in G . Hence (2) implies (1), and (4) implies (3). Assume that G is a solvable PST-group. Then every subnormal subgroup of G is S-permutable, and so NSS-permutable in G . Therefore, (5) implies (2).

Now we show that (3) implies (5). Suppose that every subgroup of $F^*(G)$ is SS-permutable in G . Then every subgroup of $F^*(G)$ is τ -quasinormal in G . By Lemma 2.3, G is a solvable PST-group, and thus (5) holds. Finally, we prove that (1) implies (4). Assume that G is solvable and every subnormal subgroup of G is SS-permutable in G . Then $F^*(G) = F(G)$

by [14, X, Corollary 13.7(d)]. Therefore, every subgroup of $F(G)$ is SS-permutable in G . By Lemma 2.1(5), every subgroup of $F(G)$ is S-permutable, and so NSS-permutable in G . Hence (4) follows. This ends the proof of Theorem B.

Proof of Theorem C. (2) implies (1) is clear. Assume that (1) holds. By Lemma 2.1(5), whenever $H \leq K$ are two p -subgroups of G with $p \in \pi(G)$, H is S-permutable in $N_G(K)$. It follows from [5, Theorem 4] that G is a solvable PST-group, and so (3) follows. By [5, Theorem 4] again, we also see that (3) implies (2).

Proof of Theorem D. Firstly we show that (1) is equivalent to (3). Clearly, (3) implies (1). Now consider that G is an SST-group. Then every subnormal subgroup of G is SS-permutable in G . By Theorem B, G is a PST-group. Let L be the nilpotent residual of G . Note that every subgroup of L is normal in G . Then for any subgroup H of G , H is SS-permutable in HL . Since HL is subnormal in G , we have that HL is SS-permutable in G . This implies H is SS-permutable in G . Hence (1) implies (3), and thus (1) and (3) are equivalent. With a similar argument as above, we get that (2) is equivalent to (4).

Next we shall prove that (5) is equivalent to (7). Clearly, (5) implies (7). Now suppose that every cyclic subgroup of G of prime power order is SS-permutable in G . Then G is a BT-group by Lemma 2.1(4) and Lemma 2.4. Let L be the nilpotent residual of G and H be a p -subgroup of G with $p \in \pi(G)$. If either $H \leq L$ or H is cyclic, then there is nothing to prove. Hence we may suppose that $p \in \pi(G) \setminus \pi(L)$ and H is not cyclic. If for every element $h \in H$, $\langle h \rangle$ is S-permutable in G , then H is S-permutable in G by [20, Corollary 1], and so H is SS-permutable in G . We may, therefore, assume that there exists an element $x \in H$ such that $\langle x \rangle$ is not S-permutable in G . By hypothesis, $\langle x \rangle$ is SS-permutable in G . Let K be an SS-permutable supplement of $\langle x \rangle$ in G . By Lemma 2.1(7), we may let $O_p(G) \leq K$. If $H \cap K = H$, then $\langle x \rangle \leq H \leq K$. This implies that $K = G$ and $\langle x \rangle$ is S-permutable in G , a contradiction. Hence $H \cap K < H$. By induction, $H \cap K$ is SS-permutable in G . Let T be an SS-permutable supplement of $H \cap K$ in G . By Lemma 2.1(7), we may also let $O_p(G) \leq T$. Now we claim that $K \cap T$ is an SS-permutable supplement of H in G . Clearly, $H(K \cap T) = \langle x \rangle(H \cap K)(K \cap T) = G$. For every Sylow q -subgroup Q of $K \cap T$ with $q \in \pi(K \cap T)$ and $p \neq q$, since G is a BT-group, we have that H permutes with Q by Theorem 1.1. Let P be a Sylow p -subgroup of $K \cap T$. Then there exist a Sylow p -subgroup K_p of K and a Sylow p -subgroup T_p of T such that $P = K_p \cap T = K \cap T_p$. By Lemma 2.7(2), $[\langle x \rangle, K_p] \leq O_p(G)$ and $[H \cap K, T_p] \leq O_p(G)$. Since $O_p(G) \leq K_p \cap T$, we have that $\langle x \rangle P = \langle x \rangle(K_p \cap T) = (K_p \cap T)\langle x \rangle = P\langle x \rangle$. Similarly, we can obtain that $(H \cap K)P = P(H \cap K)$. Therefore, $HP = \langle x \rangle(H \cap K)P = P\langle x \rangle(H \cap K) = PH$, and thus the claim follows. Then H is SS-permutable in G , and so (5) holds. With a similar discussion as above, we get that (6) is equivalent to (8).

Now we show that (3) implies (4). Assume that every subgroup of G is SS-permutable in

G . By Theorem B, G is a PST-group, and so G is supersolvable. It follows from Lemma 2.5 that every subgroup of G is NSS-permutable in G . Hence (4) holds. Evidently, (4) implies (6), and (6) implies (5). Finally, assume that every subgroup of G of prime power order is SS-permutable in G . By Lemma 2.6, it is easy to see that every subgroup of G is SS-permutable in G . This shows that (5) implies (3). The proof is thus completed.

Proof of Corollary 1.4. The corollary is directly follows from Theorem D, Lemma 2.1(4) and Theorem 1.1.

Proof of Corollary 1.6. Suppose that G is a solvable SST-group. By Theorem D and Lemma 2.1(1), every subgroup of G is a solvable SST-group. Moreover, by Lemma 2.1(2) and (3), every epimorphic image of G is also a solvable SST-group.

Proof of Corollary 1.7. By Theorem D, we have that (1) implies (3). Clearly, (3) implies (2). Then we only need to prove that (2) implies (1). Suppose that every subgroup of G is either SS-permutable or abnormal in G . By Lemma 2.1(4) and [22, Lemma 1], G is supersolvable. Let H be a p -subgroup of G with $p \in \pi(G)$. If p is not the smallest prime divisor of $|G|$, then it is easy to see that H is not abnormal in G . This induces that H is SS-permutable in G . We may, therefore, assume that p is the smallest prime divisor of $|G|$. If H is not a Sylow p -subgroup of G , then H is not abnormal in G , and thereby H is SS-permutable in G . If $H \in \text{Syl}_p(G)$, then for every $G_q \in \text{Syl}_q(G)$ with $q \in \pi(G)$ and $p \neq q$, since G_q is SS-permutable in G , H permutes with G_q . Let K be the Hall p' -subgroup of G . Then K is an SS-permutable supplement of H in G , and so H is SS-permutable in G . Hence every subgroup of G of prime power order is SS-permutable in G . By Theorem D, G is an SST-group. This shows that (2) implies (1).

Proof of Theorem E. Note that for every maximal subgroup L_1 of L and every maximal subgroup D_1 of D , $L_1 \rtimes D$ and $L \rtimes D_1$ are maximal subgroups of G . If $\Phi(G) \cap L \not\leq \Phi(L)$, then L has a maximal subgroup L_1 such that $\Phi(G) \cap L \not\leq L_1$. It follows that $\Phi(G) \cap L \leq L_1 D \cap L \leq L_1$, a contradiction. Hence $\Phi(G) \cap L \leq \Phi(L)$. With a similar discussion as above, we get that $\Phi(G) \cap D \leq \Phi(D)$.

Since $L \trianglelefteq G$, we have that $\Phi(L) \leq \Phi(G)$. Consequently, $\Phi(G) \cap L = \Phi(L)$. Now we claim that $\Phi(P) \leq \Phi(G)$ for every Sylow p -subgroup P of G with $p \in \pi(D)$. As G is a solvable SST-group, $\Phi(P)$ is SS-permutable in G by Theorem D. Let K be an SS-permutable supplement of $\Phi(P)$ in G . Then $P = P \cap \Phi(P)K = \Phi(P)(P \cap K)$. This implies that $P \leq K$, and so $K = G$. Hence $\Phi(P)$ is S-permutable in G . By [20, Lemma A], $O^p(G) \leq N_G(\Phi(P))$. It follows that $\Phi(P) \trianglelefteq G$. By [12, A, Theorem 9.2(d)], $\Phi(P) \leq \Phi(G)$, and thus the claim holds. Since D is a nilpotent Hall subgroup of G , we have that $\Phi(D) \leq \Phi(G)$, and so $\Phi(G) \cap D = \Phi(D)$. Consequently, $\Phi(G) = (\Phi(G) \cap L) \times (\Phi(G) \cap D) = \Phi(L) \times \Phi(D)$. Note that G is supersolvable

and for every Sylow subgroup P of G , $\Phi(P) \leq \Phi(G)$. This implies that every Sylow subgroup of $G/\Phi(G)$ is elementary abelian. Then by [13, Theorem 2], $G/\Phi(G)$ is a complemented group.

Proof of Theorem F. Firstly we show that (1) implies (2). Suppose that G is an SST-group. Then by Theorem D, every subgroup of G of prime power order is SS-permutable in G . For every p -subgroup P of G with $p \in \pi(G) \setminus \pi(L)$, let K be an SS-permutable supplement of P in G and K_p be a Sylow p -subgroup of K . By Lemma 2.1(8), we have that $PK_p \in Syl_p(G)$. It follows from Lemma 2.1(6) and Lemma 2.7(2) that $[H, K_p^l] \leq O_p(G)$ for every $l \in L$, and so $[H, \langle K_p^L \rangle] \leq O_p(G)$. Thus (2) follows.

It is obvious that (2) implies (3). Now we prove that (3) implies (1). Assume that (3) holds. Then by [12, A, Lemma 7.3(c)], $[x^n, \langle K_p^L \rangle] = [x, \langle K_p^L \rangle]^{x^{n-1}} \cdots [x, \langle K_p^L \rangle]^x [x, \langle K_p^L \rangle] \leq O_p(G)$ for all $n \in \mathbb{N}$. Hence $[\langle x \rangle, \langle K_p^L \rangle] \leq O_p(G)$. By Lemma 2.8, $\langle x \rangle$ is SS-permutable in G . Note that every subgroup of L is normal in G . Therefore, every cyclic subgroup of G of prime power order is SS-permutable in G . By Theorem D, G is an SST-group, and thereby (1) holds. This completes the proof of Theorem F.

Proof of Theorem G. By Theorem 1.1 and [11, Corollary 2.3], the necessity is obvious. Now we prove the sufficiency. By induction, we may assume that $n = 2$. By [11, Corollary 2.3] again, $G_1 \times G_2$ is a solvable PST-group. Evidently, the nilpotent residual of $G_1 \times G_2$ is $L_1 \times L_2$. Let $\{p, q\} \subseteq \pi(G_1 \times G_2) \setminus \pi(L_1 \times L_2)$ and $p \neq q$. For every Sylow p -subgroup G_p of $G_1 \times G_2$, there exist a Sylow p -subgroup G_{1p} of G_1 and a Sylow p -subgroup G_{2p} of G_2 such that $G_p = G_{1p} \times G_{2p}$. Similarly, for every Sylow q -subgroup G_q of $G_1 \times G_2$, there exist a Sylow q -subgroup G_{1q} of G_1 and a Sylow q -subgroup G_{2q} of G_2 such that $G_q = G_{1q} \times G_{2q}$. Since G_1 and G_2 are solvable BT-groups, by Theorem 1.1, we have that $[G_{1p}, G_{1q}] = 1$ and $[G_{2p}, G_{2q}] = 1$. It follows that $[G_p, G_q] = [G_{1p}G_{2p}, G_{1q}G_{2q}] = 1$. Therefore, by Theorem 1.1 again, $G_1 \times G_2$ is a solvable BT-group.

Proof of Theorem H. By induction, we may assume that $n = 2$. Let L_1 and L_2 be the nilpotent residuals of G_1 and G_2 , respectively. By Corollary 1.4 and Theorem G, $G_1 \times G_2$ is a solvable BT-group. Obviously, the nilpotent residual of $G_1 \times G_2$ is $L_1 \times L_2$. Let P be a p -subgroup of G with $p \in \pi(G_1 \times G_2) \setminus \pi(L_1 \times L_2)$. Without loss of generality, we may suppose that $p \in \pi(G_1) \setminus \pi(L_1)$ for $(|G_1|, |G_2|) = 1$. By Theorem F, G_1 has a p -subgroup K_p such that $PK_p \in Syl_p(G_1)$ and $[P, \langle K_p^{L_1} \rangle] \leq O_p(G_1)$. It follows that G has a p -subgroup K_p such that $PK_p \in Syl_p(G)$ and $[P, \langle K_p^{L_1 \times L_2} \rangle] \leq O_p(G)$. By Theorem F again, $G_1 \times G_2$ is a solvable SST-group.

The following lemma is the main step in the proof of Theorem I.

Lemma 3.1. Let G be a solvable PST-group, and $Z_\infty(G)$ denote the hypercenter of G . If $G/Z_\infty(G)$ is a solvable SST-group (resp. BT-group), then G is a solvable SST-group (resp. BT-group).

BT-group).

Proof. Let L be the nilpotent residual of G and D be a system normalizer of G . Firstly we prove the lemma for BT-groups. Suppose that $G/Z_\infty(G)$ is a solvable BT-group. By [12, I, Theorem 5.9(b)], $Z_\infty(G) \leq D$. Note that $LZ_\infty(G)/Z_\infty(G)$ is the nilpotent residual of $G/Z_\infty(G)$. Let $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$ with $\{p, q\} \subseteq \pi(D)$ and $p \neq q$. By Theorem 1.1, $[G_p, G_q] \leq Z_\infty(G)$. We may, therefore, assume that $G_p G_q Z_\infty(G) \leq D$. This implies that $[G_p, G_q] = 1$. By Theorem 1.1, G is a solvable BT-group.

Now we prove the lemma for SST-groups. Suppose that $G/Z_\infty(G)$ is a solvable SST-group. Then by above and Corollary 1.4, G is a solvable BT-group. Let P be a p -subgroup of G with $p \in \pi(G) \setminus \pi(L)$. By Theorem D, $PZ_\infty(G)/Z_\infty(G)$ is SS-permutable in $G/Z_\infty(G)$. Let $K/Z_\infty(G)$ be an SS-permutable supplement of $PZ_\infty(G)/Z_\infty(G)$ in $G/Z_\infty(G)$. Then K is a supplement of P in G and for every $K_p \in \text{Syl}_p(K)$, we have that $PK_p Z_\infty(G) = K_p P Z_\infty(G)$. We may, therefore, assume that $PK_p Z_\infty(G) \leq D$. Let D_p be the Sylow p -subgroup of D . By Lemma 2.7(1), $[P, K_p] \leq D_p' \trianglelefteq G$. Therefore, $[P, K_p] \leq O_p(G)$. By [12, I, Theorem 5.9(b)], $O_p(G) \leq D_G = Z_\infty(G) \leq K$, and thereby $O_p(G) \leq K_p$. This implies that $PK_p = K_p P$. By Theorem 1.1, P permutes with every Sylow q -subgroup of K with $q \in \pi(K)$ and $q \neq p$. Hence P is SS-permutable in G . Note that every subgroup of L is normal in G . Therefore, every subgroup of G of prime power order is SS-permutable in G . By Theorem D, G is a solvable SST-group.

Proof of Theorem I. The necessity is obvious. We only need to prove the sufficiency. Let L be the nilpotent residual of G and D be a system normalizer of G . By [9, Theorem B], G is a solvable PST-group. Note that the class of solvable SST-groups (resp. BT-groups) is closed under taking epimorphic images. Then it is easy to see that $G/Z_\infty(G)$ satisfies the hypothesis of the theorem. If $Z_\infty(G) \neq 1$, then by induction, $G/Z_\infty(G)$ is a solvable SST-group (resp. BT-group). It follows from Lemma 2.9 that G is a solvable SST-group (resp. BT-group). We may, therefore, assume that $Z_\infty(G) = 1$. By Lemma 2.7(1) and [12, I, Theorem 5.9(b)], we have that $D' \leq D_G = Z_\infty(G) = 1$. This implies that $G' = LD' = L$. Hence $N'' \leq G'' = 1$, and so G is a solvable SST-group (resp. BT-group).

References

- [1] R. K. Agrawal. Finite groups whose subnormal subgroups permute with all Sylow subgroups. *Proc. Amer. Math. Soc.*, 47(1):77–83, 1975.
- [2] M. J. Alejandro, A. Ballester-Bolinchés, and M. C. Pedraza-Aguilera. Finite soluble groups with permutable subnormal subgroups. *J. Algebra*, 240:705–722, 2001.

- [3] A. Ballester-Bolinches, J. C. Beidleman, and H. Heineken. A local approach to certain classes of finite groups. *Comm. Algebra*, 31(12):5931–5942, 2003.
- [4] A. Ballester-Bolinches and R. Esteban-Romero. Sylow permutable subnormal subgroups of finite groups II. *Bull. Austral. Math. Soc.*, 64:479–486, 2001.
- [5] A. Ballester-Bolinches and R. Esteban-Romero. Sylow permutable subnormal subgroups of finite groups. *J. Algebra*, 251:727–738, 2002.
- [6] A. Ballester-Bolinches and R. Esteban-Romero. On finite soluble groups in which Sylow permutability is a transitive relation. *Acta Math. Hungar.*, 101(3):193–202, 2003.
- [7] A. Ballester-Bolinches, R. Esteban-Romero, and M. Asaad. *Products of finite groups*. Walter de Gruyter, Berlin/New York, 2010.
- [8] J. C. Beidleman and H. Heineken. Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups. *J. Group Theory*, 6:139–158, 2003.
- [9] J. C. Beidleman, H. Heineken, and M. F. Ragland. Solvable PST-groups, strong Sylow bases and mutually permutable products. *J. Algebra*, 321:2022–2027, 2009.
- [10] Z. Chen. On a theorem of Srinivasan (in Chinese). *J. Southwest Normal Univ. Nat. Sci.*, 12(1):1–4, 1987.
- [11] J. Chifman. Note on direct products of certain classes of finite groups. *Comm. Algebra*, 37:1831–1842, 2009.
- [12] K. Doerk and T. Hawkes. *Finite Soluble Groups*. Walter de Gruyter, Berlin/New York, 1992.
- [13] P. Hall. Complemented groups. *J. London Math. Soc.*, 12:201–204, 1937.
- [14] B. Huppert and N. Blackburn. *Finite groups III*. Springer-Verlag, Berlin/Heidelberg, 1982.
- [15] O. H. Kegel. Sylow-gruppen und subnormalteiler endlicher gruppen. *Math. Z.*, 78:205–221, 1962.
- [16] S. Li, Z. Shen, J. Liu, and X. Liu. The influence of SS-quasinormality of some subgroups on the structure of finite groups. *J. Algebra*, 319:4275–4287, 2008.
- [17] V. O. Lukyanenko and A. N. Skiba. On weakly τ -quasinormal subgroups of finite groups. *Acta Math. Hungar.*, 125(3):237–248, 2009.
- [18] V. O. Lukyanenko and A. N. Skiba. Finite groups in which τ -quasinormality is a transitive relation. *Rend. Semin. Mat. Univ. Padova*, 124:231–246, 2010.
- [19] D. J. S. Robinson. *A course in the theory of groups*. Springer-Verlag, New York, 1996.
- [20] P. Schmid. Subgroups permutable with all Sylow subgroups. *J. Algebra*, 207:285–293, 1998.
- [21] L. Wang, Y. Li, and Y. Wang. Finite groups in which (S-)semipermutability is a transitive relation. *Int. J. Algebra*, 2(3):143–152, 2008.

- [22] Q. Zhang. s -semipermutability and abnormality in finite groups. *Comm. Algebra*, 27(9):4515–4524, 1999.